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Lawrence Oriented Matroids and a Problem of McMullen on Projective Equivalences of Polytopes

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Consider the following question introduced by McMullen: Determine the largest integer $n = f(d)$ such that any set of n points in general position in the affine d-space R^d can be mapped by a projective transformation onto the vertices of a convex polytope. It is known that $2d+1 \leq f(d) < (d+1)(d+2)/2$ and it is conjectured that $f(d) = 2d+1$. In this paper, we show that $f(d) < 2d + \lceil \frac{d+1}{2} \rceil$ by using a well-known oriented matroid generalization of the above problem.

1. INTRODUCTION

The following question has been introduced by McMullen see [6]:

[P1] Determine the largest integer $n = f(d)$ such that for any given n points in general position in affine d-space R^d there is a projective transformation mapping these points onto the vertices of a convex polytope.

Larman [6] has shown that $2d + 1 \leq f(d) \leq (d+1)^2$ for any $d \geq 2$ and conjectured that the lower bound is tight, i.e. $f(d) = 2d + 1$. Moreover, he has shown the validity of this conjecture for $d = 2, 3$. Las Vergnas [7] gave a better upper bound by showing that $f(d) < (d+1)(d+2)/2$. Forge *et al.*[4] have proved Larman's conjecture for $d = 4$. In this paper, we obtain the following improved general bound.

THEOREM 1.1. *For $d \geq 4$ we have $f(d) < 2d + \lceil \frac{d+1}{2} \rceil$. In other words, there exists a set of $2d + \lceil \frac{d+1}{2} \rceil$ points in general position in R^d which is not projectively equivalent to the set of vertices of a d-polytope.*

A generalization of [P1] in terms of oriented matroids has been proposed by Cordovil and da Silva [2]:

[P2] Determine the largest integer $n = g(r)$ such that for any uniform rank r oriented matroid $\mathcal{M}$ there is an acyclic reorientation of $\mathcal{M}$ without interior points.

The reader is referred to [1] for definitions and properties of oriented matroids. [P1] is the particular case of [P2] obtained by considering the rank $d+1$ acyclic oriented matroid of the given n points in R^d. Now a permissible projective transformation on n points in R^d corresponds to an acyclic reorientation of $\mathcal{M}$ and conversely, see [2]. Hence, we have $r = d + 1$ and $g(r) \leq f(r+1)$.

In order to prove Theorem 1.1, we shall construct a representable oriented matroid $\mathcal{M}$ of rank $r \geq 3$ with $2(r-1) + \lceil \frac{r}{2} \rceil$ elements such that any acyclic reorientation of $\mathcal{M}$ has at least one interior point.

Our method uses special oriented matroids called *Lawrence oriented matroids* which are unions of rank one uniform oriented matroids. Indeed, we construct particular Lawrence oriented matroids with the required properties. The idea to explore the class of Lawrence oriented matroids in relation to the McMullen problem was originally suggested by Las Vergnas (personal communication). Lawrence oriented matroids have nice properties; for instance, Roudneff and Sturmfels [9] gave an inductive proof that a Lawrence oriented matroid of rank r on n elements has exactly n r-simplices.

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In Section 2, we study and give general results concerning Lawrence oriented matroids needed for the rest of the paper. The results given in this section are also proved in [5] where a problem about reconstruction of arrangements is studied. Nevertheless, we include the proofs here not only for completeness but also for a better understanding of Lawrence oriented matroids and their acyclic reorientations together with the corresponding interior points that play an important, if not crucial, role in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1.

2. Lawrence Oriented Matroids

A *Lawrence oriented matroid* $\mathcal{M}$ of rank r on the totally ordered set $E = \{1, \dots, n\}, r \leq n$, denotes any uniform oriented matroid obtained as the union of r uniform oriented matroids $\mathcal{M}_1, \dots, \mathcal{M}_r$ of rank 1 on $(E, <)$ (see [8, 9]).

We can also define the Lawrence oriented matroids via the signature of their bases, that is via their chirotope χ. Indeed, the chirotope χ corresponds to some Lawrence oriented matroid $\mathcal{M}_A$ if and only if there exists a matrix $A = (a_{i,j})\ 1 \leq i \leq r, 1 \leq j \leq n$ with entries from $\{+1, -1\}$ (where the ith row corresponds to the chirotope of the oriented matroid $\mathcal{M}_i$) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i} \qquad (*)$$

where B is an ordered r-tuple $j_1 \leq \cdots \leq j_r$ elements of E.

Remarks. Let $A = (a_{i,j})\ 1 \leq i \leq r, 1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$ and $\mathcal{M}_A$ its corresponding Lawrence oriented matroid.

(i) The coefficients $a_{i,j}$ with $i > j$ or $j - n > i - r$ do not play any role in the definition of $\mathcal{M}_A$ (since they never appear in $(*)$). So, we may give them any arbitrary value from $\{+1, -1\}$ or ignore them completely.
(ii) The opposite chirotope $-\chi$ is obtained by inversing the sign of all the coefficients of a line of A.
(iii) The oriented matroid ${}_{\bar{c}}\mathcal{M}_A$ is obtained by inversing the sign of all the coefficients of column c in A.

Let us introduce right away a new notation for a better understanding of the reorientation classes of Lawrence oriented matroids. Let $A = (a_{i,j})\ 1 \leq i \leq r, 1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$. We define the *chess board* $B[A]$ of size $(r-1) \times (n-1)$, where square $s(i, j)$ is signed $+1$ or -1 according to the product given by $a_{i,j}a_{i+1,j}a_{i,j+1}a_{i+1,j+1}$ for each $i = 1, \dots, r-1$ and $j = 1, \dots, n-1$. We may colour square $s(i, j)$ *white* if its corresponding sign is $+1$ or *black* otherwise.

Let ${}_{\bar{k}}A$ be the matrix obtained from A by inversing the sign of all the coefficients of column k in A. Then square $s(i, j)$ is white in $B[{}_{\bar{k}}A]$ if and only if $s(i, j)$ is white in $B[A]$. That is, the board is invariant under inversing the signs of all the coefficients of a given column of A. Therefore, if E and F are matrices such that $\mathcal{M}_E$ and $\mathcal{M}_F$ are in the same orientation class, then $B[E] = B[F]$. We will use the chess board description to prove our main theorem in Section 3.

We shall construct the *Top Travel* $[TT]$ (and the *Bottom Travel* $[BT]$) on the entries of A, formed by horizontal and vertical movements according to the following procedure, see [5].

Procedure

(1) TT (BT) starts at $a_{1,1}$ (at $a_{r,n}$)
(2) Suppose that TT (BT) arrives at $a_{i,j}$. Let s (s') be the minimum (maximal) integer $j < s \leq n$ $(1 \leq s' < j)$ such that $a_{i,j} = -a_{i,s}$ $(a_{i,j} = -a_{i,s'})$.
(3) **If** s (s') does not exist then
TT goes (horizontally) to $a_{i,n}$ and stops
(BT goes (horizontally) to $a_{i,1}$ and stops)
(4) **else**

(a) **If** $1 \leq i \leq r-1$ $(n \leq i \leq 2)$ then
TT goes (horizontally) to $a_{i,s}$ and then goes (vertically) to $a_{i+1,s}$
(BT goes (horizontally) to $a_{i,s'}$ and then goes (vertically) to $a_{i-1,s'}$)
(b) **else**
TT goes (horizontally) to $a_{r,s}$ and stops
(BT goes (horizontally) to $a_{1,s'}$ and stops)

We write $TT = (a_{1,1}, \ldots, a_{1,t_1}, a_{2,t_1}, \ldots, a_{2,t_2}, \ldots, a_{x,t_{x-1}}, \ldots, a_{x,t_x})$, $1 \leq x \leq r+1$, where $a_{l,t_{l-1}}, \ldots, a_{l,t_l}$ are the entries in line l of A' along TT with $1 \leq l \leq x \leq r+1$. We may also use the shorter notation $TT = [t_0, \ldots, t_1][t_1, \ldots, t_2] \ldots [t_{x-1}, \ldots, t_x]$ where $[t_{l-1}, \ldots, t_l]$ denote the entries $[a_{l,t_{l-1}}, \ldots, a_{l,t_l}]$ with $1 \leq l \leq x \leq r+1$ and $t_0 = 1$ (similar for BT).

Recall that an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is *acyclic* if it does not contain positive circuits (otherwise, $\mathcal{M}$ is called *cyclic*). We say that an element $e \in E$ of a uniform oriented acyclic matroid is *interior* if there exists a signed circuit $C = (C^+, C^-)$ with $C^- = \{e\}$. It is equivalent to define the interior points as the elements whose reorientation give a cyclic matroid.

LEMMA 2.1. *Let $A = (a_{i,j})$ $1 \leq i \leq r$, $1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$ and let TT and BT be the Top and Bottom Travels constructed on A respectively. Then the following conditions are equivalent*

(a) $\mathcal{M}_A$ is cyclic
(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$
(c) BT ends at $a_{1,s'}$ for some $1 < s' \leq n$.

PROOF. We only show the equivalence between (a) and (b) (the equivalence between (a) and (c) is analogous). Let $C = (j_1, \ldots, j_{r+1})$, $j_1 \leq \cdots \leq j_{r+1}$ be a circuit of $\mathcal{M}_A$. Given a sign of element $j_1 = +1$ (or -1) of C and the chirotope χ of $\mathcal{M}_A$, we find the sign $C(j_l)$ of element j_l of C, $2 \leq l \leq r+1$ as follows: $C(j_{i+1}) = -C(j_i)\chi(C \setminus j_i)\chi(C \setminus j_{i+1})$ for $i = 1, \ldots, r$. Hence, by $(*)$ we have $C(j_{i+1}) = -C(j_i)a_{i,j_i}a_{i,j_{i+1}}$.

Without loss of generality, assume the entry $a_{1,1}$ to be $+1$. Suppose that TT ends at $a_{r,s}$ for some $1 \leq s < n$. Then, there exist exactly r columns $j_1, \ldots, j_r$ such that $a_{i,j_{i+1}} = -a_{i,j_i}$ for $i = 1, \ldots, r-1$ (TT makes a vertical movement in the first $r-1$).

Thus, by the above formula, $C(1, j_1, \ldots, j_r)$ is a positive circuit.

Now, suppose that $\mathcal{M}_A$ is cyclic. Then there exists a positive circuit $C(j_0, j_1, \ldots, j_r)$ with $a_{i,j_{i-1}} = -a_{i,j_i}$ for $i = 1, \ldots, r$. Thus, TT meets line $i+1$ for the first time at column l for some $l \leq j_i$ and the result follows. In fact, if $\mathcal{M}$ is cyclic, then the set of columns where TT (BT) makes a vertical movement defines the first (the last) positive circuit in the lexicographical order. □

We denote by ${}_{\bar{k}}TT$ and ${}_{\bar{k}}BT$ the corresponding travels of ${}_{\bar{k}}A$. We say that TT and BT are *parallel* at column k with $2 \le k \le n-1$ in A if $TT = (a_{1,1}, \dots, a_{i,k-1}, a_{i,k}, a_{i,k+1}, \dots)$ and either $BT = (a_{r,n}, \dots, a_{i,k+1}, a_{i,k}, a_{i,k-1}, \dots)$ or $BT = (a_{r,n}, \dots, a_{i+1,k+1}, a_{i+1,k}, a_{i+1,k-1}, \dots)$, $1 \le i \le r$.

LEMMA 2.2. *Let $A = (a_{i,j})$ $1 \le i \le r$, $1 \le j \le n$ be a matrix with entries from $\{+1, -1\}$ such that $\mathcal{M}_A$ is acyclic and let TT and BT be the travels in A. Then k is an interior element of $\mathcal{M}_A$ if and only if*

(a) $BT = (a_{r,n}, \dots, a_{1,2}, a_{1,1})$ for $k = 1$,
(b) $TT = (a_{1,1}, \dots, a_{r,n-1}, a_{r,n})$ for $k = n$,
(c) TT and BT are parallel at k for $2 \le k \le n-1$.

PROOF. We say that TT (resp. BT) arrives at line $r+1$ (resp. line 0) if TT (resp. BT) ends in step 4(b) of the above procedure. By Lemma 2.2, k is an interior element of $\mathcal{M}_A$ if and only if TT (or BT) does not arrive at line $r+1$ (or 0) in A but ${}_{\bar{k}}TT$ (or ${}_{\bar{k}}BT$) does in ${}_{\bar{k}}A$. Since $\mathcal{M}_A$ is acyclic, TT and BT do not arrive at lines $r+1$ and 0 in A respectively. We shall show, in each case, that either ${}_{\bar{k}}TT$ (or ${}_{\bar{k}}BT$) arrives at line $r+1$ (or 0) in ${}_{\bar{k}}A$.

Part (a) (resp. (b)) clearly follows, since ${}_{\bar{1}}BT$ (resp. ${}_{\bar{n}}TT$) arrives at line 0 (resp. $r+1$) in ${}_{\bar{1}}A$ (resp. ${}_{\bar{n}}A$) if and only if $BT = (a_{r,n}, \dots, a_{1,2}, a_{1,1})$ (resp. $TT = (a_{1,1}, \dots, a_{r,n-1}, a_{r,n})$).

For part (c), let $TT = [t_0, \dots, t_1][t_1, \dots, t_2] \dots [t_{l-1}, \dots, t_l]$, with $1 \le l \le r$ and let $P_{TT} = \{a_{i,j} \mid j > t_i \text{ for each } i = 1, \dots, l\}$, that is, P_{TT} is the set of all the entries which are exactly above TT. Let $BT_{a_{i,k}}$ be the travel in A, starting at $a_{i,k}$ and formed according to the above rules for BT. Consider the travel ${}_{\bar{k}}BT_{a_{i,k}}$ in ${}_{\bar{k}}A$. We claim that (1) if either $a_{i,k} \in P_{TT}$ or $a_{i,k} \in [t_{i-1}, \dots, t_i]$ with $k > t_{i-1}$ for some $1 \le i \le l$, then all elements of ${}_{\bar{k}}BT_{a_{i,k}}$ belong to either P_{TT} or TT and (2) ${}_{\bar{k}}BT_{a_{i,k}}$ arrives at line 0. Indeed, part (1) can be easily verified and part (b) follows since ${}_{\bar{k}}BT_{a_{i,j}}$ arrives at $a_{1,m}$ for some $t_1 \le m$ (by part (1)) and $a_{1,t_1} = -a_{1,t_1-1}$.

Let us suppose that $TT = (a_{1,1}, \dots, a_{i,k-1}, a_{i,k}, a_{i,k+1}, \dots)$ and $BT = (a_{r,n}, \dots, a_{j,k+1}, a_{j,k}, a_{j,k-1}, \dots)$ with $1 \le i \le j \le r$. Clearly, ${}_{\bar{k}}BT$ can be seen as BT from $a_{r,n}$ until $a_{j,k}$ followed by $a_{j,k-1}, a_{j-1,k-1}$ and ${}_{\bar{k}}BT_{a_{j-1,k-1}}$. So, after reorienting column k, we have that $a_{j-1,k-1} \in PP_{TT}$ (or $a_{j-1,k-1} \in [t_{i-1}, \dots, t_i]$ with $k > t_{i-1}$) if and only if $j = i$ (or $j = i+1$). Hence, by the above claim, k is an interior element if and only if TT and BT are parallel at column k, $2 \le k \le n-1$. □.

EXAMPLE. Let E be the following matrix with entries in $\{+1, -1\}$ (we just keep the signs for a shorter notation).

$$E = \begin{pmatrix} + & + & + & - & + & - & - & + & - \\ + & + & - & + & - & - & + & + & + \\ - & - & - & + & + & + & + & - & + \\ + & - & + & - & + & + & + & + & + \end{pmatrix}.$$

The Top and Bottom Travels in E' are shown in Figure 1.

It is easy to check, by Lemma 2.2 and Figure 1, that 1, 6, 7 and 9 are interior elements of the oriented matroid $\mathcal{M}_E$.

3. THE UPPER BOUND

Define a *plain travel* T on the entries of A, formed by horizontal (from left to right) and vertical (from top to bottom) movements such that (a) T starts with $a_{1,1}, a_{1,2}$, (b) T cannot make two consecutive vertical movements and (c) T ends at $a_{i,n}$ for $1 \le i \le r$.

$$
\begin{array}{c|ccccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\
\text{TT} & + & + & + & - & + & - & - & + & - & \\
 & + & + & - & + & - & - & + & + & + & \\
 & - & - & - & + & + & + & + & - & + & \\
 & + & - & + & - & + & + & + & + & + & \text{BT}
\end{array}
$$

FIGURE 1. Top and Bottom Travels on E'.

Let $N(A)$ be the number of all plain travels in matrix A. It is known that the number of acyclic reorientations of an oriented matroid $\mathcal{M}$ of rank r on n elements, $0 < r < n$, is given by $\frac{1}{2}t(\underline{\mathcal{M}}; 2, 0)$ where $t(\underline{\mathcal{M}}; x, y)$ denotes the *Tutte polynomial* of the underlying matroid of $\mathcal{M}$, see [1]. In the case of the uniform oriented matroid $\mathcal{M}_A$ we have that,

$$
\begin{aligned}
t(\underline{\mathcal{M}_A}; 2, 0) &= \sum_{B \subseteq E} (-1)^{|B|-r(B)} = \sum_{i=0}^{r} \binom{n}{i} + \sum_{i=r+1}^{n} (-1)^{i-r(M)} \binom{n}{i} \\
&= \sum_{i=0}^{n} \binom{n}{i} - 2\left[\binom{n}{r+1} + \binom{n}{r+3} + \cdots + \binom{n}{s}\right]
\end{aligned}
$$

where $s = n$ if $n - r$ is odd or $s = n - 1$ otherwise. Since $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ then

$$
\begin{aligned}
t(\underline{\mathcal{M}_A}; 2, 0) &= 2\sum_{i=0}^{n-2} \binom{n-1}{i} + \binom{n-1}{n-1} + \binom{n}{n} - 2\left[\sum_{i=r}^{n-2} \binom{n-1}{i} + \binom{s}{s}\right] \\
&= 2\sum_{i=0}^{r-1} \binom{n-1}{i} + 2 - 2 = 2N(A).
\end{aligned}
$$

We give a constructive interpretation of the latter equality.

LEMMA 3.1. *Let $A = (a_{i,j})$, $1 \le i \le r$, $1 \le j \le n$ be a matrix with entries from $\{+1, -1\}$. Then the above procedure gives a natural bijection between the set of all plain travels of A and the set of all acyclic reorientations of $\mathcal{M}_A$.*

PROOF. Let T be a plain travel in A and let $\bar{A}$ be the (unique) matrix obtained from A such that T is transformed into TT of $\bar{A}$. This can be done by inversing the sign of a set of columns of A. Note that, by Remarks (i) and (iii), $\mathcal{M}_{\bar{A}}$ remains in the same orientation class as $\mathcal{M}_A$. Since T ends at line i with $1 \le i \le r$ then TT also does and, by Lemma 2.1, T corresponds to an acyclic reorientation of $\mathcal{M}_A$. Moreover the set of reoriented elements of the acyclic reorientation is given by the corresponding set of columns which signs were inversed in order to transform T into TT.

Let T' be a plain travel in A different from T. Suppose that T and T' are the same plain travel just until entry $a_{i,j}$ and then they move to different entries. We have either of the following cases.

(1) T carries on with entries $a_{i,j+1}, a_{i,j+2}$ and T' carries on with entries $a_{i,j+1}, a_{i+1,j+1}$. If $a_{i,j} = a_{i,j+1}$ the sign of column $j + 1$ was not reversed (resp. was reversed) in

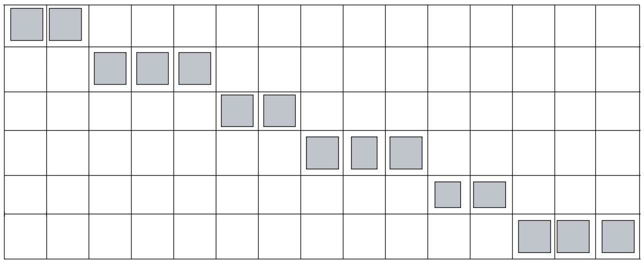

FIGURE 2. $B[A^*_{7,16}]$.

A in order to transform T (resp. T') into the Top Travel in $\bar{A}$. So, the corresponding reorientations are different. If $a_{i,j} = -a_{i,j+1}$ the sign of column $j+1$ was reversed (resp. was not reversed) in A in order to transform T (resp. T') into the Top Travel in $\bar{A}$. So, the corresponding reorientations are also different.

(2) T carries on with entries $a_{i,j+1}, a_{i+1,j+1}$ and T' carries on with entries $a_{i,j+1}, a_{i,j+2}$. Analogous to case (1). □

In order to prove Theorem 1.1, it is thus sufficient to construct a matrix A of size $r \times 2(r-1) + \lceil \frac{r}{2} \rceil$, $r \geq 3$ such that for any given plain travel of A the corresponding Top Travel in $\bar{A}$ (as defined in Lemma 3.1) has at least one interior element. To do this, we define a special chess board.

DEFINITION 3.2. Let $A^* = A^*_{r,2(r-1)+\lceil \frac{r}{2} \rceil} = (a^*_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq 2(r-1) + \lceil \frac{r}{2} \rceil$, $r \geq 2$ be the matrix with entries from $\{+1, -1\}$ such that this gives rise to the board $B[A^*]$ with $s(k,l)$ black for each $k = 1, \ldots, r-1$ and either $l = \frac{5}{2}k - 2$, $\frac{5}{2}k - 1$ and $\frac{5}{2}k - 2$ if k is even or $l = \frac{1}{2}(5k-3)$ and $\frac{1}{2}(5k-1)$ if k is odd. The rest of the squares in $B[A^*]$ being white, see Figure 2.

Notice that there might exist several matrices giving rise to the above chess board (all belonging to the same orientation class).

REMARKS. Given a plain travel in A^*, let TT^* and BT^* be the corresponding Top and Bottom Travels in $\bar{A}^*$ (recall that $B[A^*] = B[\bar{A}^*]$).

(I) Suppose that TT^* passes throughout $a^*_{i,j}$, $a^*_{i,j+1}$ and $a^*_{i,j+2}$ with $1 \leq i \leq r-1$ and $1 \leq j \leq n-3$ (that is, $a^*_{i,j}$ and $a^*_{i,j+1}$ have the same sign) and that BT^* arrives at $a^*_{k,j}$ for some $i \leq k \leq r$.
   (a) If $s(l,j)$ is white for all $i \leq l \leq k-1$ then BT^* goes to $a^*_{k,j-1}$.
   (b) If $s(l,j)$ is black for some $i \leq l \leq k-1$ and $s(m,j)$ is white for all $i \leq m \leq k-1$, $m \neq l$ then BT^* goes to $a^*_{k,j-1}$ and $a^*_{k-1,j-1}$

(II) Suppose that TT^* passes throughout $a^*_{i,j}$, $a^*_{i,j+1}$ and $a^*_{i+1,j+1}$ with $1 \leq i \leq r-1$ and $1 \leq j \leq n-1$ (that is, $a^*_{i,j}$ and $a^*_{i,j+1}$ have a different sign) and that BT^* arrives at $a^*_{k,j}$ for some $i+1 \leq k \leq r$.
   (a) If $s(l,j)$ is white for all $i \leq l \leq k-1$ then BT^* goes to $a^*_{k,j-1}$ and $a^*_{k-1,j-1}$.
   (b) If $s(l,j)$ is black for some $i \leq l \leq k-1$ and $s(m,j)$ is white for all $i \leq m \leq k-1$, $m \neq l$ then BT^* goes to $a^*_{k,j-1}$.

We say that element $a^*_{m,2(m-1)+\lceil \frac{m}{2} \rceil}$ is a 2-*corner* (resp. 3-*corner*) of A^*, $1 \le m \le r-1$ if m is odd (resp. if m is even), that is, if $s(m, 2(m-1)+\lceil \frac{m}{2} \rceil)$ and $s(m, 2(m-1)+\lceil \frac{m}{2} \rceil+1)$ (resp. $s(m, 2(m-1)+\lceil \frac{m}{2} \rceil)$, $s(m, 2(m-1)+\lceil \frac{m}{2} \rceil+1)$ and $s(m, 2(m-1)+\lceil \frac{m}{2} \rceil+2)$) are black. We call element $a^*_{m-1,2(m-1)+\lceil \frac{m}{2} \rceil}$, $2 \le m \le r-1$ an *above-corner*.

CLAIM 3.3. *Let TT^* and BT^* be the Top and Bottom Travels in $A^* = A^*_{r,2(r-1)+\lceil \frac{r}{2} \rceil}$, $r \ge 3$ respectively. Suppose that TT^* starts with $a^*_{1,1}$, $a^*_{1,2}$ and $a^*_{1,3}$ and that it never meets any other corner or above-corner after column* 3. *Then BT^* arrives at either $a^*_{2,3}$ or $a^*_{1,3}$.*

PROOF. We use induction on r and the above remarks. It is true for $r = 3$. Suppose it is true for $r = k-1$ and we shall show it for $r = k \ge 4$. Notice first that if BT^* arrives at corner $a^*_{m,2(m-1)+\lceil \frac{m}{2} \rceil}$ with $6 \le m \le k-1$ then the result follows by the induction hypothesis. Moreover, if BT^* arrives at element $a^*_{i,2(m-1)+\lceil \frac{m}{2} \rceil}$ with $6 \le m \le k-1$ and $i \le m$ then, by the above remarks, BT^* would arrive at either element at $a^*_{1,3}$ or at corner $a^*_{m',2(m'-1)+\lceil \frac{m}{2} \rceil}$ for some $2 \le m' < m$ (and the result follows again by the induction hypothesis).

So, we assume that BT^* always passes strictly below all corners after column 3. Suppose that BT^* ends at $a^*_{i,3}$ for some $3 \le i \le k$, that is, BT^* makes at most $k-3$ vertical movements. Now, TT^* makes exactly $2(k-1)+\lceil \frac{k}{2} \rceil - 3$ horizontal movements and at most $k-1$ vertical movements, from right to left, to arrive at $a^*_{1,3}$. We know, by the rules of construction of TT^*, that for each vertical movement we must also count one horizontal movement. So, TT^* makes at least $2(k-1)+\lceil \frac{k}{2} \rceil - 3 - (k-1) = k-4+\lceil \frac{k}{2} \rceil$ (single) horizontal movements, from right to left, until $a^*_{1,3}$ is attained. Since BT^* is always below the corners then, by the above remarks, for each (single) horizontal movement of TT^*, BT^* makes a vertical movement. So, BT^* makes at least $k-4+\lceil \frac{k}{2} \rceil$ vertical movements. But $k-4+\lceil \frac{k}{2} \rceil > k-3$ for each $k \ge 3$ which is a contradiction. □

CLAIM 3.4. *Let TT^* and BT^* be the Top and Bottom Travels in $A^* = A^*_{r,2(r-1)+\lceil \frac{r}{2} \rceil}$, $r \ge 4$ respectively. Suppose that TT^* passes throughout $a^*_{2,6}$ and that it never meets any other corner or above-corner after column* 6. *Then BT^* arrives at either $a^*_{2,6}$ or $a^*_{3,6}$ or $a^*_{4,6}$.*

PROOF. We use induction on r. The result is clearly true for $r = 4$. Suppose it is true for $r = k-1$ and we shall show it for $r = k \ge 5$. By similar arguments as in Claim 3.3, we may assume that BT^* always passes strictly below all corners after column 6. Since TT^* does not meet any corner after column 6 then it passes throughout $a^*_{2,8}$. Moreover, TT^* passes throughout $a^*_{3,j}$ and $a^*_{2,j}$ for some $9 \le j$. Now, if TT^* would have passed throughout $a^*_{3,8}$ then, by Claim 3.3, BT^* would have passed throughout either $a^*_{3,8}$ or $a^*_{4,8}$. Hence, when TT^* meets $a^*_{2,8}$ BT^*, arrives at either $a^*_{4,8}$ or $a^*_{5,8}$ and the result follows. □

We may now prove Theorem 1.1.

PROOF OF THEOREM 1.1. We shall show that for any given plain travel in $A^* = A^*_{r,2(r-1)+\lceil \frac{r}{2} \rceil}$, $r \ge 3$, the corresponding Top Travel TT^* in $\bar{A}^* = \bar{A}^*_{r,2(r-1)+\lceil \frac{r}{2} \rceil}$ (where $\bar{A}^*$ is defined as in the proof of Lemma 3.1) has at least one interior element (or equivalently any acyclic reorientation of $\mathcal{M}_{A^*}$ has at least one interior element). We shall use the fact that $B[A^*] = B[\bar{A}^*]$.

We use induction on r. The result is clearly true for $r = 3$. Assume it is true for $r = k-1$ and we show it for $r = k$. Now, suppose that $a^*_{m,2(m-1)+\lceil \frac{m}{2} \rceil}$, $1 \le m \le k-1$ is the last corner that TT^* meets. If $m = 1$ then, by Claim 3.3, BT^* arrives at either $a^*_{2,3}$ (hence 1 is an interior element) or $a^*_{1,3}$ (hence 1 and 2 are interior elements) and the result follows (similar for the case $m = 2$). So, suppose that $2 < m \le k-1$. We have two cases.

Case A. If $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil}$ is a 2-corner, then TT^* must pass through elements $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+1}$ and $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+2}$ (otherwise, TT^* would meet another corner after column m). Now, we apply Claim 3.3 to the submatrix obtained from of A^* by removing rows $1,\ldots,m-1$ and columns $1,\ldots,2(m-1)+\lceil\frac{m}{2}\rceil-1$ and conclude that BT^* arrives at either $a^*_{m+1,2(m-1)+\lceil\frac{m}{2}\rceil+2}$ (in such case BT^* goes to $a^*_{m+1,2(m-1)+\lceil\frac{m}{2}\rceil+1}$, $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+1}$, $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil}$ and the result follows by induction) or $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+2}$ (in such case BT^* goes to $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+1}$, $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil}$ and $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+1}$ is an interior element).

Case B. If $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil}$ is a 3-corner then TT^* must pass through element $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+1}$, $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+2}$ and $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+3}$ (same reason as in case A). Now, we apply Claim 3.4 to the submatrix obtained from of A^* by removing rows $1,\ldots,m-2$ and columns $1,\ldots,2(m-1)+\lceil\frac{m}{2}\rceil-3$ and conclude that BT^* arrives at either $a^*_{m,2(m-1)+\lceil\frac{m}{2}\rceil+3}$ or $a^*_{m+1,2(m-1)+\lceil\frac{m}{2}\rceil+3}$ or $a^*_{m+2,2(m-1)+\lceil\frac{m}{2}\rceil+3}$ and the result follows by using similar arguments as in case A (that is, in each case, we use the induction hypothesis or find an interior element). $\square$

4. Concluding Remarks

Lemmas 2.1 and 2.2 give a very simple algorithm (linear on the number of elements) to verify if a reorientation of a Lawrence oriented matroid has interior elements. Indeed, it suffices to construct the Top and Bottom Travels in the corresponding matrix and their interior elements. This is very helpful since one should normally check all the $\binom{n}{r}$ circuits to find the interior points. D. Forge and M. Las Vergnas (private communication) have used this fact to try to find a Lawrence oriented matroid $\mathcal{M}$ of rank 5 (resp. 6) with 10 (resp. 12) elements such that any acyclic reorientation of $\mathcal{M}$ has at least one interior element. Unfortunately, after exploring all non-isomorphic Lawrence oriented matroids with the above parameters, they did not find any with the required property, see [3]. We believe that the above upper bound cannot be much improved by using Lawrence oriented matroids.

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